A multi-agent matching problem

- Measure the distance between two sets of $N$ agents $x^{(0)} := \{x^{(0)}_i\}_{i=1}^N$ and $x^{(1)} := \{x^{(1)}_j\}_{j=1}^N$ as the solution to the matching problem
  \[
  d(x^{(0)}, x^{(1)}) := \min_{\phi \in \text{perm}} \sum_{i=1}^N \|x^{(0)}_i - x^{(1)}_{\phi(i)}\|_2^2,
  \]
  where perm denotes the set of all permutations of the index set $\{1, \ldots, N\}$.

- Define $C \in \mathbb{R}^{N \times N}$ as $C_{ij} = \|x^{(0)}_i - x^{(1)}_j\|_2^2$. Then the matching distance is the solution to the optimal transport problem
  \[
  d(x^{(0)}, x^{(1)}) = \min_{M \in \mathbb{R}_{+}^{N \times N}} \langle C, M \rangle
  \]
  subject to
  \[
  M1 = 1, \quad M^T1 = 1,
  \]

Figure 1: Illustration of matching distance between two distributions of agents.
Control interpretation

- The matching problem can be interpreted as a minimum-energy control problem, since

\[ C_{ij} = \|x_i(0) - x_j(1)\|_2^2 = \min_{u \in L_2([0,1])} \int_0^1 \|u(t)\|_2^2 dt \]

subject to \( \dot{x}(t) = u(t), \quad x(0) = x_i(0), \quad x(1) = x_j(1) \),

- If each agent is governed by the dynamics \( \dot{x}(t) = Ax(t) + Bu(t) \), where \((A, B)\) is controllable, define the cost

\[ C_{ij}^{A,B} = \min_{u \in L_2([0,1])} \int_0^1 \|u(t)\|_2^2 dt \]

subject to \( \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_i(0), \quad x(1) = x_j(1) \),

- Using this cost in the optimal transport problem yields the matching problem with dynamics

\[ d_{A,B}(x(0), x(1)) = \min_{\phi \in \text{perm}} \sum_{i=1}^N \int_0^1 \|u_i(t)\|_2^2 dt \]

subject to \( \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad t \in [0, 1], \quad x_i(0) = x_i(0), \quad x_i(1) = x_i(1) \), for \( i = 1, \ldots, N \).

- Displacement interpolation:

\[ \min_{x^{(\tau)}, \tau = 1, \ldots, T-1} \sum_{\tau=1}^T d_{A,B}(x^{(\tau-1)}, x^{(\tau)}) \]

- Steering an ensemble of agents with dynamics \( \dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t) \) using displacement interpolation:
State estimation interpretation

- Consider a set of agents governed by the stochastic dynamical systems

\[ dx_i(t) = Ax_i(t)dt + Bd\xi_i(t), \quad i = 1, \ldots, N, \]

where \( \xi_i \), for \( i = 1, \ldots, N \), are independent Brownian motions.

- **Theorem:** Given these dynamics, the maximum likelihood estimation problem of finding the most likely pairing between the unpaired state observations \( x_i(0) = x_i^{(0)} \) and \( x_i(1) = x_i^{(1)} \) is given by \( d_{A,B}(x^{(0)}, x^{(1)}) \).

- Filtering: Assume output measurements are described by the discrete-time processes

\[ y_i(\tau) = Cx_i(\tau) + D\omega_i(\tau), \]

where \( \omega_i(\tau) \) are independent Gaussian random variables.

Then, a maximum likelihood estimation for the hidden states is

\[
\minimize_{x(\tau), \tau = 0, \ldots, T} \sum_{\tau = 1}^{T} d_{A,B}(x^{(\tau-1)}, x^{(\tau)}) + \sum_{\tau = 0}^{T} d_{C,D}(x^{(\tau)}, y^{(\tau)}).
\]

- Example: Tracking of 7 agents from noisy and incomplete observations. Consider the dynamics

\[ \dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t) \]

with observations \( y_i(\tau) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i(\tau) + \begin{bmatrix} 1 \end{bmatrix} \omega_i(\tau) \), where \( u_i \) and \( \omega_i \) have variance 10\(^{-2}\). The ground truth is plotted as full lines, observations are marked by circles, and the estimates are plotted as dashed lines.

![Graphical representation of state estimation](image.png)

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