Control and Estimation of Ensembles via Structured Optimal Transport

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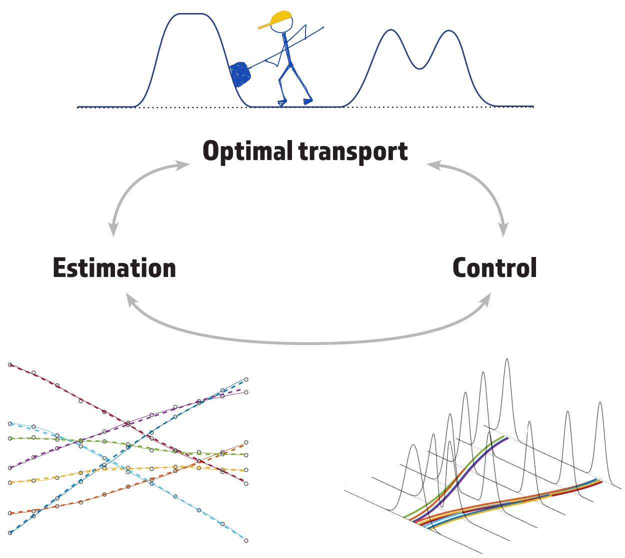


Full paper: `https://ieeexplore.ieee.org/abstract/document/9491021`

A multi-agent matching problem

- Measure the distance between two sets of N agents $x^{(0)} := \{x_i^{(0)}\}_{i=1}^N$ and $x^{(1)} := \{x_j^{(1)}\}_{j=1}^N$ as the solution to the matching problem

$$d(x^{(0)}, x^{(1)}) := \underset{\phi \in \text{perm}}{\text{minimize}} \sum_{i=1}^{N} \|x_i^{(0)} - x_{\phi(i)}^{(1)}\|_2^2,$$

  where perm denotes the set of all permutations of the index set $\{1, \dots, N\}$.

- Define $C \in \mathbb{R}^{N\times N}$ as $C_{ij} = \|x_i^{(0)} - x_j^{(1)}\|_2^2$. Then the matching distance is the solution to the *optimal transport problem*

$$\begin{aligned} d(x^{(0)}, x^{(1)}) \;=\; & \underset{M\in\mathbb{R}_+^{N\times N}}{\text{minimize}} && \langle C, M\rangle \\ & \text{subject to} && M\mathbf{1} = \mathbf{1}, \\ & && M^T\mathbf{1} = \mathbf{1}, \end{aligned}$$

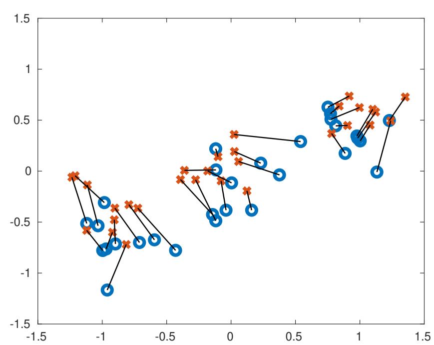


Figure 1: Illustration of matching distance between two distributions of agents.

Control interpretation

- The matching problem can be interpreted as a minimum-energy control problem, since

$$
\begin{aligned}
C_{ij} = \|x_i^{(0)} - x_j^{(1)}\|_2^2 = \ \underset{u \in L_2([0,1])}{\text{minimize}} \ & \int_0^1 \|u(t)\|_2^2 dt \\
\text{subject to} \ & \dot{x}(t) = u(t), \\
& x(0) = x_i^{(0)}, \quad x(1) = x_j^{(1)},
\end{aligned}
$$

- If each agent is governed by the dynamics $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$, where $(\mathcal{A}, \mathcal{B})$ is controllable, define the cost

$$
\begin{aligned}
C_{ij}^{\mathcal{A},\mathcal{B}} = \underset{u \in L_2([0,1])}{\text{minimize}} \ & \int_0^1 \|u(t)\|^2 dt \\
\text{subject to} \ & \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \\
& x(0) = x_i^{(0)}, \quad x(1) = x_j^{(1)},
\end{aligned}
$$

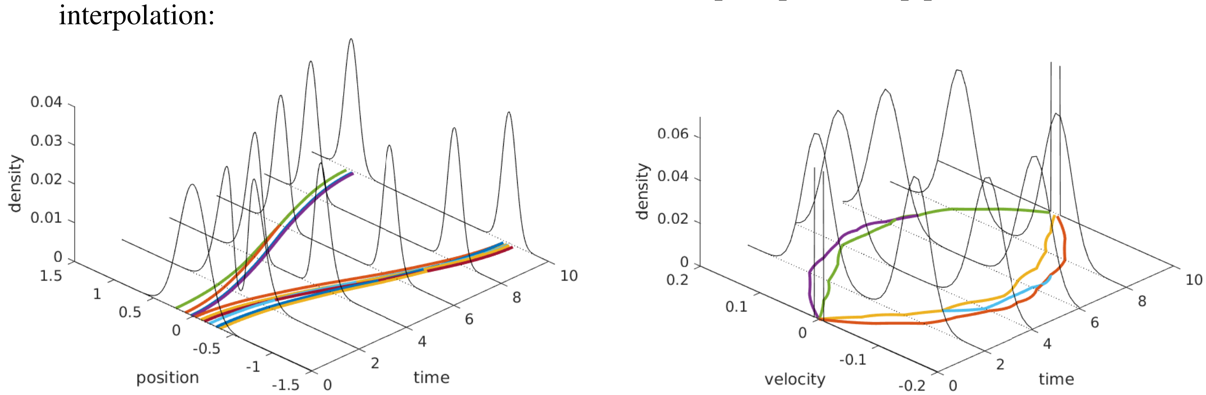
- Using this cost in the optimal transport problem yields the matching problem with dynamics

$$
\begin{aligned}
d_{\mathcal{A},\mathcal{B}}(x^{(0)}, x^{(1)}) = \underset{\substack{\phi \in \text{perm} \\ u_i \in L_2([0,1]), \\ i=1,\dots,N}}{\text{minimize}} \ & \sum_{i=1}^{N} \int_0^1 \|u_i(t)\|_2^2 dt \\
\text{subject to} \ & \dot{x}_i(t) = \mathcal{A}x_i(t) + \mathcal{B}u_i(t), \quad t \in [0,1], \\
& x_i(0) = x_i^{(0)}, \ x_i(1) = x_{\phi(i)}^{(1)}, \quad \text{for } i = 1, \dots, N.
\end{aligned}
$$

- Displacement interpolation:

$$
\underset{x^{(\tau)}, \tau=1,\dots,\mathcal{T}-1}{\text{minimize}} \quad \sum_{\tau=1}^{\mathcal{T}} d_{\mathcal{A},\mathcal{B}}(x^{(\tau-1)}, x^{(\tau)})
$$

- Steering an ensemble of agents with dynamics $\dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t)$ using displacement interpolation:



State estimation interpretation

- Consider a set of agents governed by the stochastic dynamical systems

$$dx_i(t) = \mathcal{A}x_i(t)dt + \mathcal{B}d\xi_i(t), \quad i = 1, \ldots, N,$$

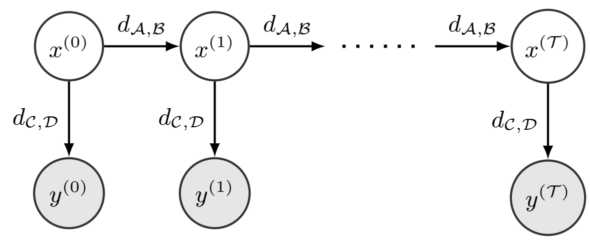
  where ξ_i, for $i = 1, \ldots, N$, are independent Brownian motions.

- **Theorem:** *Given these dynamics, the maximum likelihood estimation problem of finding the most likely pairing between the unpaired state observations* $x_i(0) = x_i^{(0)}$ *and* $x_i(1) = x_i^{(1)}$ *is given by* $d_{\mathcal{A},\mathcal{B}}(x^{(0)}, x^{(1)})$.

- Filtering: Assume output measurements are described by the discrete-time processes

$$y_i(\tau) = \mathcal{C}x_i(\tau) + \mathcal{D}\omega_i(\tau),$$

  where $\omega_i(\tau)$ are independent Gaussian random variables.
  Then, a maximum likelihood estimation for the hidden states is



$$\underset{x^{(\tau)},\tau=0,\ldots,\mathcal{T}}{\text{minimize}} \quad \sum_{\tau=1}^{\mathcal{T}} d_{\mathcal{A},\mathcal{B}}\left(x^{(\tau-1)}, x^{(\tau)}\right) + \sum_{\tau=0}^{\mathcal{T}} d_{\mathcal{C},\mathcal{D}}\left(x^{(\tau)}, y^{(\tau)}\right).$$

- Example: Tracking of 7 agents from noisy and incomplete observations. Consider the dynamics $\dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t)$ with observations $y_i(\tau) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i(\tau) + \begin{bmatrix} 1 \end{bmatrix} \omega_i(\tau)$, where u_i and ω_i have variance 10^{-2}. The ground truth is plotted as full lines, observations are marked by circles, and the estimates are plotted as dashed lines.